An Algebraic Theory for Shared-State Concurrency

Yotam Dvir [1] Ohad Kammar [2] Ori Lahav [1]

[1]Tel Aviv University

[2]University of Edinburgh

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Goal

To fit [shared-state] concurrency semantics on equal footing
with other semantic models of computational effects

Method

Using the standard algebraic effects approach, in which we

define a denotational semantics

over a monad

representing an equational theory

Virtues of Algebraic Effects

What's so good about this approach?

Compositionality As in any denotational semantics

Higher-order The language supports higher-order functions "out-of-the-box"

Uniformity General results / similar proof techniques

Modularity Combine equational theories, e.g. (global-state + yield) $\oplus$ non-determinism

Comparability Easy to compare different languages / semantics, e.g. Abadi & Plotkin

Abstraction Program behaviour analysis using the monad and the equations

Implementability Monads are ubiquitous in functional programming

Rundown

One slide summary of this talk

Programming Language – Standard operational-semantics

Higher-Order $\lambda x.\, M$ (Call-by-Value)

Shared-State: Assignment $:=$, Dereference $?$, Interleaving concurrency $\|$

Denotational Semantics

Standard Monadic [Moggi] $[\![NM]\!]\gamma := [\![N]\!]\gamma \gg\!= \lambda f.\, [\![M]\!]\gamma \gg\!= f$

Based on Traces [Brookes] $\langle \left(\begin{smallmatrix} a & b & c \\ 1 & 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b & c \\ 1 & 0 & 1 \end{smallmatrix}\right) \rangle \langle \left(\begin{smallmatrix} a & b & c \\ 0 & 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b & c \\ 1 & 1 & 0 \end{smallmatrix}\right) \rangle\, 1 \in [\![\mathsf{a} := \mathsf{c}?\, ;\, \mathsf{a}?]\!]$

Algebraic Effects [Plotkin, Power, Hyland, Levy]:

(global-state + yield) $\oplus$ non-determinism $[\![\mathsf{a} := 1]\!] = \mathrm{U}_{\mathsf{a},1}\langle\rangle \vee \mathrm{U}_{\mathsf{a},1}\mathrm{Y}\langle\rangle$

Ordered by non-determinism $t \leq t \vee s$

Theorem (Adequacy for shared-state)

$$[\![M]\!] \leq [\![N]\!] \implies \forall C[-].\, \sigma, C[M] \rightsquigarrow\!* \rho, V \implies \sigma, C[N] \rightsquigarrow\!* \rho, V$$

Section 1

Story Time

Contextual Equivalence

Programs $\quad \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{b}\mathbf{?} \mathbin{\mathbf{;}} \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{c}\mathbf{?} \quad$ and $\quad \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{c}\mathbf{?} \quad$ execute identically

Fact

$$\begin{array}{cccc} \sigma, & \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{b}\mathbf{?} \mathbin{\mathbf{;}} \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{c}\mathbf{?} & \rightsquigarrow * & \rho, \quad V \\ & & \Updownarrow & \\ \sigma, & \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{c}\mathbf{?} & \rightsquigarrow * & \rho, \quad V \end{array}$$

Example

$$\begin{array}{cccc} \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{pmatrix}, & \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{b}\mathbf{?} \mathbin{\mathbf{;}} \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{c}\mathbf{?} & \rightsquigarrow * & \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{pmatrix}, \quad \langle \rangle \\ \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{pmatrix}, & \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{c}\mathbf{?} & \rightsquigarrow * & \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{pmatrix}, \quad \langle \rangle \end{array}$$

$\mathrm{Locs} = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\} \qquad \mathrm{Vals} = \{0, 1\}$ $\qquad$ e.g. $\quad \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{pmatrix} : \mathrm{Locs} \to \mathrm{Vals}$

Contextual Equivalence

Does this mean they are interchangeable?

For all program contexts $C[-]$?

$$\begin{array}{rrcll} \sigma, & C[\mathsf{a} := \mathsf{b}?\, ;\, \mathsf{a} := \mathsf{c}?] & \rightsquigarrow^* & \rho, & V \\ & & \Updownarrow & & \\ \sigma, & C[\mathsf{a} := \mathsf{c}?] & \rightsquigarrow^* & \rho, & V \end{array}$$

Example (Sequential Context)

$$\begin{array}{rrcll} \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{pmatrix}, & \mathsf{a} := \mathsf{b}?\, ;\, \mathsf{a} := \mathsf{c}?\, ;\, \mathsf{a}? & \rightsquigarrow^* & \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{pmatrix}, & 1 \\ \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{pmatrix}, & \mathsf{a} := \mathsf{c}?\, ;\, \mathsf{a}? & \rightsquigarrow^* & \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{pmatrix}, & 1 \end{array}$$

$\mathrm{Locs} = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$ $\quad$ $\mathrm{Vals} = \{0, 1\}$ $\qquad$ e.g. $\begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{pmatrix} : \mathrm{Locs} \to \mathrm{Vals}$

Contextual Equivalence

Does this mean they are interchangeable?

For all program contexts $C[-]$?

$$\begin{array}{lrcll} \sigma, & C[\mathsf{a} := \mathsf{b}\mathbf{?}\,;\,\mathsf{a} := \mathsf{c}\mathbf{?}] & \rightsquigarrow^* & \rho, & V \\ & & \Updownarrow & & \\ \sigma, & C[\mathsf{a} := \mathsf{c}\mathbf{?}] & \rightsquigarrow^* & \rho, & V \end{array}$$

Example (Concurrent Context)

$$\begin{array}{lrcll} \left(\begin{smallmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{smallmatrix}\right), & \mathsf{a} := \mathsf{b}\mathbf{?}\,;\,\mathsf{a} := \mathsf{c}\mathbf{?} \parallel \mathsf{a}\mathbf{?} & \rightsquigarrow^* & \left(\begin{smallmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{smallmatrix}\right), & \langle\langle\rangle, 0\rangle \\ \left(\begin{smallmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{smallmatrix}\right), & \mathsf{a} := \mathsf{c}\mathbf{?} \parallel \mathsf{a}\mathbf{?} & \not\rightsquigarrow^* & \left(\begin{smallmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{smallmatrix}\right), & \langle\langle\rangle, 0\rangle \end{array}$$

$\mathrm{Locs} = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$ $\quad \mathrm{Vals} = \{0, 1\}$ $\qquad$ e.g. $\left(\begin{smallmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{smallmatrix}\right) : \mathrm{Locs} \to \mathrm{Vals}$

Global-State (Sequential) Denotational Semantics

[Plotkin, Power]

$$\text{(LU-comm)} \qquad \mathrm{L}_{\ell}\left(v.\ \mathrm{U}_{\ell',w} x_v\right) = \mathrm{U}_{\ell',w} \mathrm{L}_{\ell}\left(v.\ x_v\right) \quad \ell \neq \ell'$$

$$\begin{aligned}
[\![\mathsf{a} := \mathsf{b}?\,;\,\mathsf{a} := \mathsf{c}?]\!] &= \mathrm{L}_{\mathsf{b}}\left(v.\ \mathrm{U}_{\mathsf{a},v} \mathrm{L}_{\mathsf{c}}\left(w.\ \mathrm{U}_{\mathsf{a},w}\langle\rangle\right)\right) \\
\text{(LU-comm)} \longrightarrow &= \mathrm{L}_{\mathsf{b}}\left(v.\ \mathrm{L}_{\mathsf{c}}\left(w.\ \mathrm{U}_{\mathsf{a},v} \mathrm{U}_{\mathsf{a},w}\langle\rangle\right)\right) \\
\text{(UU-last)} \longrightarrow &= \mathrm{L}_{\mathsf{b}}\left(v.\ \mathrm{L}_{\mathsf{c}}\left(w.\ \mathrm{U}_{\mathsf{a},w}\langle\rangle\right)\right) \\
\text{(L-noop)} \longrightarrow &= \mathrm{L}_{\mathsf{c}}\left(w.\ \mathrm{U}_{\mathsf{a},w}\langle\rangle\right) = [\![\mathsf{a} := \mathsf{c}?]\!]
\end{aligned}$$

Fact (Adequacy for global-state [folklore])

$$[\![M]\!] = [\![N]\!] \implies \forall\, C[-].\ \sigma, C[M] \rightsquigarrow^* \rho, V \iff \sigma, C[N] \rightsquigarrow^* \rho, V$$

Thus a := b? ; a := c? and a := c? are interchangeable

Compiler can optimize a := b? ; a := c? $\twoheadrightarrow$ a := c? (or the other direction too)

Extends to Shared-State (Concurrency)?

[Hyland, Levy, Plotkin, Power]

$$[\![M \parallel N]\!] = ?$$

Add an operator (choice): $\vee$

Add axioms: $\mathrm{U}_{\mathrm{a},v}(t \vee s) = \mathrm{U}_{\mathrm{a},v} t \vee \mathrm{U}_{\mathrm{a},v} s \;\; ; \;\; t \vee s = s \vee t \; \ldots$

Define partial-order: $t \leq t \vee s$

Desired (Adequacy for shared-state)

$$[\![M]\!] \leq [\![N]\!] \implies \forall C[-].\, \sigma, C[M] \rightsquigarrow\!\ast\; \rho, V \implies \sigma, C[N] \rightsquigarrow\!\ast\; \rho, V$$

Justifies the opposite $N \twoheadrightarrow M$ (no new results = no bugs)

Example (from before)

$$\left(\begin{smallmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ 1 & 0 & 1 \end{smallmatrix}\right), \quad \mathrm{a} := \mathrm{b}? \,;\, \mathrm{a} := \mathrm{c}? \parallel \mathrm{a}? \quad \rightsquigarrow\!\ast \quad \left(\begin{smallmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ 1 & 0 & 1 \end{smallmatrix}\right), \quad \langle\langle\rangle, 0\rangle$$

$$\left(\begin{smallmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ 1 & 0 & 1 \end{smallmatrix}\right), \quad \mathrm{a} := \mathrm{c}? \parallel \mathrm{a}? \quad \cancel{\rightsquigarrow\!\ast} \quad \left(\begin{smallmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ 1 & 0 & 1 \end{smallmatrix}\right), \quad \langle\langle\rangle, 0\rangle$$

Contradicts $[\![\mathrm{a} := \mathrm{b}? \,;\, \mathrm{a} := \mathrm{c}?]\!] \leq [\![\mathrm{a} := \mathrm{c}?]\!]$ – Not adequate

What Should Change to Fix This?

$$[\![\mathsf{a} := \mathsf{b?} \, ; \mathsf{a} := \mathsf{c?}]\!] \overset{!}{\neq} [\![\mathsf{a} := \mathsf{c?}]\!]$$

~~Invalidate equations~~ OR Change denotations

$$\begin{aligned}[\![\mathsf{a} := \mathsf{b?} \, ; \mathsf{a} := \mathsf{c?}]\!] &\neq \mathrm{L}_{\mathsf{b}}\left(v.\ \mathrm{U}_{\mathsf{a},v}\mathrm{L}_{\mathsf{c}}\left(w.\ \mathrm{U}_{\mathsf{a},w}\langle\rangle\right)\right) \\ &= \mathrm{L}_{\mathsf{c}}\left(w.\ \mathrm{U}_{\mathsf{a},w}\langle\rangle\right) \neq [\![\mathsf{a} := \mathsf{c?}]\!]\end{aligned}$$

get to keep global-state

Theory of Resumptions

[Hyland, Levy, Plotkin, Power]

$$\llbracket \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{b}\mathbf{?} \mathbin{\mathbf{;}} \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{c}\mathbf{?} \rrbracket \neq \mathrm{L}_{\mathsf{b}}\left(v.\ \mathrm{U}_{\mathsf{a},v}\mathrm{L}_{\mathsf{c}}\left(w.\ \mathrm{U}_{\mathsf{a},w}\langle\rangle\right)\right)$$
$$= \mathrm{L}_{\mathsf{c}}\left(w.\ \mathrm{U}_{\mathsf{a},w}\langle\rangle\right) \neq \llbracket \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{c}\mathbf{?} \rrbracket$$

Add an operator yield: Y

Add an axiom: $\mathrm{Y}(t \vee s) = \mathrm{Y}t \vee \mathrm{Y}s$

Define possible yield: $\mathrm{Y}^{?}t \coloneqq t \vee \mathrm{Y}t \quad (\mathrm{Y}^{?}t \geq t)$

$$\begin{aligned}
\llbracket \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{b}\mathbf{?} \mathbin{\mathbf{;}} \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{c}\mathbf{?} \rrbracket &= \mathrm{L}_{\mathsf{b}}\left(v.\ \mathrm{Y}^{?}\mathrm{U}_{\mathsf{a},v}\mathrm{Y}^{?}\mathrm{L}_{\mathsf{c}}\left(w.\ \mathrm{Y}^{?}\mathrm{U}_{\mathsf{a},w}\mathrm{Y}^{?}\langle\rangle\right)\right) \\
&\geq \mathrm{L}_{\mathsf{b}}\left(v.\ \mathrm{Y}^{?}\mathrm{U}_{\mathsf{a},v} \quad \mathrm{L}_{\mathsf{c}}\left(w. \quad \mathrm{U}_{\mathsf{a},w}\mathrm{Y}^{?}\langle\rangle\right)\right) \\
\ldots(\text{like before})\ldots &= \mathrm{L}_{\mathsf{c}}\left(w.\ \mathrm{Y}^{?}\mathrm{U}_{\mathsf{a},w}\mathrm{Y}^{?}\langle\rangle\right) = \llbracket \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{c}\mathbf{?} \rrbracket
\end{aligned}$$

$\mathsf{a} \mathrel{\mathbf{:=}} \mathsf{b}\mathbf{?} \mathbin{\mathbf{;}} \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{c}\mathbf{?} \twoheadrightarrow \mathsf{a} \mathrel{\mathbf{:=}} \mathsf{c}\mathbf{?}$ justified by Adequacy

Lingering Questions

We should have $[\![\mathsf{a} \mathbin{\mathbf{:=}} \mathsf{b}\mathbf{?} \mathbin{\mathbf{;}} \mathsf{a} \mathbin{\mathbf{:=}} \mathsf{c}\mathbf{?}]\!] \neq [\![\mathsf{a} \mathbin{\mathbf{:=}} \mathsf{c}\mathbf{?}]\!]$ for adequacy

Still undefined: $[\![M \parallel N]\!] = ?$

What about "pure" fragment, e.g. $[\![\mathbf{if}\ M\ \mathbf{then}\ N\ \mathbf{else}\ N]\!] \stackrel{?}{=} [\![M \mathbin{\mathbf{;}} N]\!]$

Section 2

The Algebraic-Effects Roadmap

1 Story Time

2 The Algebraic-Effects Roadmap
- Equational Theory
- Monadic Model
- Denotations
- Adequacy

3 Final Words...

Subsection 1

Equational Theory

Axioms – Global-State

[Plotkin, Power]

UL-det	$\mathrm{U}_{\ell,w}\mathrm{L}_{\ell}\left(v.\ x_v\right) = \mathrm{U}_{\ell,w}x_w$	
UU-last	$\mathrm{U}_{\ell,v}\mathrm{U}_{\ell,w}x = \mathrm{U}_{\ell,w}x$	
LU-noop	$\mathrm{L}_{\ell}\left(v.\ \mathrm{U}_{\ell,v}x\right) = x$	
LL-diag	$\mathrm{L}_{\ell}\left(v.\ \mathrm{L}_{\ell}\left(w.\ x_{v,w}\right)\right) = \mathrm{L}_{\ell}\left(v.\ x_{v,v}\right)$	
UU-comm	$\mathrm{U}_{\ell,v}\mathrm{U}_{\ell',w}x = \mathrm{U}_{\ell',w}\mathrm{U}_{\ell,v}x$	$\ell \neq \ell'$
LU-comm	$\mathrm{L}_{\ell}\left(v.\ \mathrm{U}_{\ell',w}x_v\right) = \mathrm{U}_{\ell',w}\mathrm{L}_{\ell}\left(v.\ x_v\right)$	$\ell \neq \ell'$
LL-comm	$\mathrm{L}_{\ell}\left(v.\ \mathrm{L}_{\ell'}\left(w.\ x_{v,w}\right)\right) = \mathrm{L}_{\ell'}\left(w.\ \mathrm{L}_{\ell}\left(v.\ x_{v,w}\right)\right)$	$\ell \neq \ell'$

Axioms – Non-Determinism and Yield

[Hyland, Levy, Plotkin, Power]

ND-return	$\bigvee_{\imath<1} x = x$	
ND-epi	$\bigvee_{\jmath<\beta} x_\jmath = \bigvee_{\imath<\alpha} x_{\varphi\imath}$	$\varphi : \alpha \twoheadrightarrow \beta$
ND-join	$\bigvee_{\imath<\alpha} \bigvee_{\jmath<\beta_\imath} x_{\imath,\jmath} = \bigvee_{\jmath<\sum_{\imath<\alpha}\beta_\imath} x_{\varphi\jmath}$	$\varphi : \sum_{\imath<\alpha}\beta_\imath \leftrightarrow \coprod_{\imath<\alpha}\beta_\imath$
ND-$\mathtt{L}$	$\bigvee_{\imath<\alpha} \mathrm{L}_\ell\,(v.\ x_{v,\imath}) = \mathrm{L}_\ell\,(v.\ \bigvee_{\imath<\alpha} x_{v,\imath})$	
ND-$\mathtt{U}$	$\bigvee_{\imath<\alpha} \mathrm{U}_{\ell,v} x_\imath = \mathrm{U}_{\ell,v} \bigvee_{\imath<\alpha} x_\imath$	
ND-$\mathtt{Y}$	$\bigvee_{\imath<\alpha} \mathrm{Y} x_\imath = \mathrm{Y} \bigvee_{\imath<\alpha} x_\imath$	

Subsection 2

Monadic Model

Traces – Interrupted Execution

[Brookes, Benton, Hoffman, Nigam]

$$\mathsf{a} := \mathsf{c}? \parallel \mathsf{b} := 0\, ;\, \mathsf{a}? \xrightarrow{\left\langle \left(\begin{smallmatrix} a & b & c \\ 1 & 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b & c \\ 1 & 0 & 1 \end{smallmatrix}\right) \right\rangle \left\langle \left(\begin{smallmatrix} a & b & c \\ 0 & 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b & c \\ 0 & 1 & 1 \end{smallmatrix}\right) \right\rangle \left\langle \left(\begin{smallmatrix} a & b & c \\ 0 & 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b & c \\ 1 & 1 & 1 \end{smallmatrix}\right) \right\rangle} \langle \langle\rangle, 0 \rangle$$

$$\begin{array}{llcll}
\left(\begin{smallmatrix} a & b & c \\ 1 & 0 & 1 \end{smallmatrix}\right), & \mathsf{a} := \mathsf{c}? \parallel \mathsf{b} := 0\, ;\, \mathsf{a}? & \rightsquigarrow^* & \left(\begin{smallmatrix} a & b & c \\ 1 & 0 & 1 \end{smallmatrix}\right), & \mathsf{a} := 1 \parallel \mathsf{a}? \\
 & & \wr & & \\
\left(\begin{smallmatrix} a & b & c \\ 0 & 1 & 1 \end{smallmatrix}\right), & \mathsf{a} := 1 \parallel \mathsf{a}? & \rightsquigarrow^* & \left(\begin{smallmatrix} a & b & c \\ 0 & 1 & 1 \end{smallmatrix}\right), & \mathsf{a} := 1 \parallel 0 \\
 & & \wr & & \\
\left(\begin{smallmatrix} a & b & c \\ 0 & 1 & 1 \end{smallmatrix}\right), & \mathsf{a} := 1 \parallel 0 & \rightsquigarrow^* & \left(\begin{smallmatrix} a & b & c \\ 1 & 1 & 1 \end{smallmatrix}\right), & \langle \langle\rangle, 0 \rangle
\end{array}$$

$$\mathrm{Traces}\, X := \left(\mathrm{Vals}^{\mathrm{Locs}} \times \mathrm{Vals}^{\mathrm{Locs}}\right)^{+} \cdot X$$

Example

$$\left\langle \left(\begin{smallmatrix} a & b & c \\ 1 & 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b & c \\ 1 & 0 & 1 \end{smallmatrix}\right) \right\rangle \left\langle \left(\begin{smallmatrix} a & b & c \\ 0 & 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b & c \\ 0 & 1 & 1 \end{smallmatrix}\right) \right\rangle \left\langle \left(\begin{smallmatrix} a & b & c \\ 0 & 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b & c \\ 1 & 1 & 1 \end{smallmatrix}\right) \right\rangle \langle \langle\rangle, 0 \rangle \in \mathrm{Traces}\,(\mathbf{1} \times \mathrm{Vals})$$

Model Definition

$$\underline{\mathcal{T}} X \coloneqq \mathcal{P}_{\mathrm{fin}}\left(\mathrm{Traces} X\right) \qquad \mathrm{Traces} X \coloneqq \left(\mathrm{Vals}^{\mathrm{Locs}} \times \mathrm{Vals}^{\mathrm{Locs}}\right)^{+} \cdot X$$

$$\begin{aligned}
\tilde{x} &\coloneqq \{\langle\sigma,\sigma\rangle\, x\} \\
\tilde{\mathrm{V}}_{\imath<\alpha} P_\imath &\coloneqq \bigcup_{\imath<\alpha} P_\imath \\
\tilde{\mathrm{Y}}\, P &\coloneqq \{\langle\sigma,\sigma\rangle\, \tau \mid \tau \in P\} \\
\tilde{\mathrm{L}}_\ell\,(v.\ P_v) &\coloneqq \{\langle\sigma,\rho\rangle\, \tau \mid \langle\sigma,\rho\rangle\, \tau \in P_v\ ,\ \sigma_\ell = v\} \\
\tilde{\mathrm{U}}_{\ell,v} P &\coloneqq \{\langle\sigma,\rho\rangle\, \tau \mid \langle\sigma\,[\ell \mapsto v],\rho\rangle\, \tau \in P\}
\end{aligned}$$

Theory of Resumptions axioms all hold, e.g. (UU-last): $\tilde{\mathrm{U}}_{\ell,v}\tilde{\mathrm{U}}_{\ell,w}\tilde{x} = \cdots = \tilde{\mathrm{U}}_{\ell,w}\tilde{x}$

Monad Definition

$$\mathcal{T}X \coloneqq \langle \underline{\mathcal{T}}X, \mathsf{return}, \gg\!\!= \rangle \qquad \underline{\mathcal{T}}X \coloneqq \mathcal{P}_{\mathrm{fin}}\left(\mathrm{Traces}X\right)$$

$$\begin{aligned} \mathsf{return}\, x &\coloneqq \tilde{x} \\ P \gg\!\!= f &\coloneqq \left\{ \alpha \left\langle \sigma, \varsigma \right\rangle \tau \;\middle|\; \exists \rho .\, \alpha \left\langle \sigma, \rho \right\rangle x \in P \wedge \left\langle \rho, \varsigma \right\rangle \tau \in fx \right\} \end{aligned}$$

Theorem (Representation for shared-state)

The monad $\mathcal{T}$ is equivalent to the monad induced by Res.

Corollary (Soundness & Completeness)

$$t = s \text{ (in Res.)} \iff \tilde{t} = \tilde{s} \text{ (in Model)}$$

Subsection 3

Denotations

Denotation

$$\Gamma \vdash M : A \qquad\qquad [\![M]\!] : [\![\Gamma]\!] \to \underline{\mathcal{T}}[\![A]\!]$$

Example

$$\vdash \mathsf{a} \mathbin{\mathbf{:=}} \mathsf{c}\mathbf{?} : () \qquad\qquad \mathrm{x} : \mathbf{Loc} \vdash \mathrm{x}\mathbf{?} : \mathbf{Val}$$

$$[\![\mathsf{a} \mathbin{\mathbf{:=}} \mathsf{c}\mathbf{?}]\!] : \{\!\} \to \underline{\mathcal{T}}\mathbf{1} \qquad\qquad [\![\mathrm{x}\mathbf{?}]\!] : \{\mathrm{x} \in \mathrm{Locs}\} \to \underline{\mathcal{T}}\mathrm{Vals}$$

Definition is entirely compositional:
denotation of prog. depends only on *denotations* of subparts

Denotation

$$\Gamma \vdash M : A \qquad\qquad [\![M]\!] : [\![\Gamma]\!] \to \underline{\mathcal{T}}[\![A]\!]$$

Definition is entirely compositional:
denotation of prog. depends only on *denotations* of subparts

(1/3) Standard part, including higher-order (based on [Moggi]):

$$\begin{aligned}
[\![\mathrm{x}]\!]\gamma &:= \mathsf{return}\,\gamma\mathrm{x} \\
[\![\mathbf{let}\,\mathrm{x} = M\,\mathbf{in}\,N]\!]\gamma &:= [\![M]\!]\gamma \gg\!= \lambda y.\,[\![N]\!]\,(\gamma\,[\mathrm{x} \mapsto y]) \\
[\![\langle M, N\rangle]\!]\gamma &:= [\![M]\!]\gamma \gg\!= \lambda x.\,[\![N]\!]\gamma \gg\!= \lambda y.\ \mathsf{return}\,\langle x, y\rangle \\
&\ \vdots
\end{aligned}$$

$$[\![\mathbf{if}\,M\,\mathbf{then}\,N\,\mathbf{else}\,N\,]\!] = [\![M\,\mathbf{;}\,N]\!]$$

Denotation

$$\mathit{\Gamma} \vdash M : A \qquad\qquad [\![M]\!] : [\![\mathit{\Gamma}]\!] \to \underline{\mathcal{T}}[\![A]\!]$$

Definition is entirely compositional:
denotation of prog. depends only on *denotations* of subparts

(2/3) State access part:

$$\begin{aligned} [\![M\mathbf{?}]\!]\gamma &:= [\![M]\!]\gamma \gg\!= \lambda\ell.\, \tilde{\mathrm{L}}_{\ell}\big(v.\, \tilde{\mathrm{Y}}^{?}\, \mathrm{return}\, v\big) \\ [\![M \mathbin{\mathbf{:=}} N]\!]\gamma &:= [\![M]\!]\gamma \gg\!= \lambda\ell.\, [\![N]\!]\gamma \gg\!= \lambda v.\, \tilde{\mathrm{U}}_{\ell,v}\tilde{\mathrm{Y}}^{?}\, \mathrm{return}\, \langle\rangle \end{aligned}$$

$$[\![\mathsf{a} \mathbin{\mathbf{:=}} \mathsf{b}\mathbf{?}\, \mathbf{;}\, \mathsf{a} \mathbin{\mathbf{:=}} \mathsf{c}\mathbf{?}]\!] \neq [\![\mathsf{a} \mathbin{\mathbf{:=}} \mathsf{c}\mathbf{?}]\!]$$

Denotation

$$\Gamma \vdash M : A \qquad\qquad [\![M]\!] : [\![\Gamma]\!] \to \underline{\mathcal{T}}[\![A]\!]$$

Definition is entirely compositional:
denotation of prog. depends only on *denotations* of subparts

(3/3) Concurrency part (based on [Brookes]):

$$[\![M \parallel N]\!]\gamma \quad := \quad [\![M]\!]\gamma \mathbin{|||} [\![N]\!]\gamma$$

$$(|||) : \underline{\mathcal{T}}X \times \underline{\mathcal{T}}Y \to \underline{\mathcal{T}}(X \times Y)$$

$$P \mathbin{|||} Q := \left\{ \omega \;\middle|\; \exists \tau \in P, \pi \in Q.\; \tau \parallel \pi \Longrightarrow \omega \right\}$$

Here ω is obtained by Interleaving transitions from τ and π
and sometimes Mumbling them: $\langle \sigma, \rho \rangle \langle \rho, \varsigma \rangle \mapsto \langle \sigma, \varsigma \rangle$

Formally $\tau \parallel \pi \Longrightarrow \omega$ is defined by...

Syntactic Synchronization

$$P \mathbin{|||} Q := \left\{ \omega \;\middle|\; \exists \tau \in P, \pi \in Q.\ \tau \parallel \pi \Longrightarrow \omega \right\}$$

$$\boxed{\tau \parallel \pi \Longrightarrow \omega}$$

$$\frac{}{\langle \sigma, \rho \rangle\, x \parallel \langle \rho, \varsigma \rangle\, \beta y \Longrightarrow \langle \sigma, \varsigma \rangle\, \beta\, \langle x, y \rangle}\ (\textsc{Var-Left})$$

$$\frac{\tau \parallel \pi \Longrightarrow \omega}{\langle \sigma, \rho \rangle\, \tau \parallel \pi \Longrightarrow \langle \sigma, \rho \rangle\, \omega}\ (\textsc{Brk-Left})$$

$$\frac{\tau \parallel \pi \Longrightarrow \langle \rho, \varsigma \rangle\, \omega}{\langle \sigma, \rho \rangle\, \tau \parallel \pi \Longrightarrow \langle \sigma, \varsigma \rangle\, \omega}\ (\textsc{Seq-Left})$$

VAR Interleave no more

BRK Interleave without mumbling

SEQ Interleave with mumbling

Symmetrically: (VAR-RIGHT) (BRK-RIGHT) (SEQ-RIGHT)

Subsection 4

Adequacy

Adequacy

For $\Gamma \vdash M : A$ and $\Gamma \vdash N : A$

Let $[\![M]\!] \leq [\![N]\!]$ denote $\forall \gamma \in [\![\Gamma]\!].\, [\![M]\!]\gamma \subseteq [\![N]\!]\gamma$

Theorem (Adequacy for shared-state)

$$[\![M]\!] \leq [\![N]\!] \implies \forall C[-].\, \sigma, C[M] \rightsquigarrow^* \rho, V \implies \sigma, C[N] \rightsquigarrow^* \rho, V$$

Proof via standard logical relations (higher-order "out-of-the-box")

Reward

Justify transformation $N \twoheadrightarrow M$ without simulation $(\forall C[-])$

Transformations

Standard transformations

e.g. $\textbf{if}\ M_1\ \textbf{then}\ (M_2\ \mathbf{;}\ N)\ \textbf{else}\ (M_3\ \mathbf{;}\ N) \cong (\textbf{if}\ M_1\ \textbf{then}\ M_2\ \textbf{else}\ M_3\ )\ \mathbf{;}\ N$

Proof by monad laws (same proof as with other effects!)

Redundant Access Eliminations

e.g. $\ell \mathbin{\mathbf{:=}} v\ \mathbf{;}\ \ell\mathbf{?} \twoheadrightarrow \ell \mathbin{\mathbf{:=}} v\ \mathbf{;}\ v$

Proof by mundane algebra:

$$\mathrm{U}_{\ell,v}\mathrm{Y}^{?}\mathrm{L}_{\ell}\left(w.\ \mathrm{Y}^{?}w\right) \geq \mathrm{U}_{\ell,v}\mathrm{Y}^{?}v$$

Laws of parallelism

e.g. $M \parallel N \twoheadrightarrow \langle M, N \rangle$

Proof by analysis of interleaving:

$$P \mathbin{|||} Q \supseteq P \ggg\!= \lambda x.\ Q \ggg\!= \lambda y.\ \texttt{return}\,\langle x, y\rangle$$

Redundant Read Introduction is not supported (don't have full abstraction)

e.g. $\langle\rangle \twoheadrightarrow \ell\mathbf{?}\ \mathbf{;}\ \langle\rangle$ not a consequence: $\tilde{\langle\rangle} \not\geq \tilde{\mathrm{L}}_{\ell}\left(v.\ \tilde{\mathrm{Y}}^{?}\tilde{\langle\rangle}\right)$ ("counting issue")

Section 3

Final Words...

Very Partial Related Work Timeline

1996 Brookes
- Imperative Language
- Denotation: Set of Traces

2002 Plotkin, Power
- Alg. Effects & Global-State

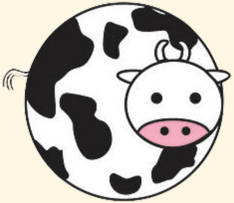
2006 Hyland, Levy, Plotkin, Power
- Non-Determinism & Resumptions (Yield)

2010 Abadi, Plotkin
- Imperative Language
- Cooperative Async. Threads
- Algebraic Effects (different semantics)

2016 Benton, Hofmann, Nigam
- Functional Language
- Monad (specified directly)

2022: Our contribution
- Standard approach, definitions and proofs
- Transformations justified algebraically
- Finite model

Extensions / Features?



Weak Memory

Type-and-Effect System

Atomic Constructs

Recursion

...

Rundown

One slide summary of this talk

Programming Language – Standard operational-semantics

Higher-Order $\lambda x.\, M$ (Call-by-Value)

Shared-State: Assignment $:=$, Dereference $?$, Interleaving concurrency $\parallel$

Denotational Semantics

Standard Monadic [Moggi] $\quad [\![NM]\!]\gamma := [\![N]\!]\gamma \gg\!= \lambda f.\, [\![M]\!]\gamma \gg\!= f$

Based on Traces [Brookes] $\quad \left\langle \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 0 & 1 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ 1 & 1 & 0 \end{pmatrix} \right\rangle 1 \in [\![\mathsf{a} := \mathsf{c}? \,;\, \mathsf{a}?]\!]$

Algebraic Effects [Plotkin, Power, Hyland, Levy]:

(global-state + yield) $\oplus$ non-determinism $\quad [\![\mathsf{a} := 1]\!] = \mathrm{U}_{\mathsf{a},1}\langle\rangle \vee \mathrm{U}_{\mathsf{a},1}\mathrm{Y}\langle\rangle$

Ordered by non-determinism $\quad t \leq t \vee s$

Theorem (Adequacy for shared-state)

$$[\![M]\!] \leq [\![N]\!] \implies \forall C[-].\, \sigma, C[M] \rightsquigarrow^{*} \rho, V \implies \sigma, C[N] \rightsquigarrow^{*} \rho, V$$