First-order Quasi-canonical Proof Systems*

Yotam Dvir and Arnon Avron

Tel-Aviv University, Israel
{yotamdvir@mail,aa@cs}.tau.ac.il

Abstract. Quasi-canonical Gentzen-type systems with dual-arity quantifiers is a wide class of proof systems. Using four-valued non-deterministic semantics, we show that every system from this class admits strong cut-elimination iff it satisfies a certain syntactic criterion of coherence. As a specific application, this result is applied to the framework of Existential Information Processing (EIP), in order to extend it from its current propositional level to the first-order one — a step which is crucial for its usefulness for handling information that comes from different sources (that might provide contradictory or incomplete information).

Keywords: Gentzen-type proof systems · cut-elimination · coherence · non-deterministic matrices · information processing · knowledge bases

Introduction

Proving the cut-elimination theorem for a given Gentzen-type system **G** is usually a difficult and detail intensive task, especially if **G** involves quantifiers that bind variables. In [3] this problem was solved for the wide class of *canonical* Gentzen-type proof systems. These are the systems in which the language features dual-arity quantifiers (i.e. quantifiers that may bind several variables and at the same time connect several formulas), and in which all the logical rules are of the ideal type which was used by Gentzen in [12]. The solution was achieved by formulating an easily checkable syntactic criterion of coherence, and showing that for canonical systems coherence is equivalent both to strong cut-elimination and to strong soundness and completeness with respect to some two-valued non-deterministic matrix. Based on results in [1], we extend this theory here to *quasi-canonical* systems, i.e. systems which are canonical 'up to negation'. (See Definitions 18 and 20 below.) Our main theorem is fairly similar to that in [3], but it is more general, and has the significant difference that the semantics we use is based on *four-valued* (rather than two-valued) non-deterministic matrices.

As a very important specific application of the general theory described above, we take the problem of gathering and processing information from a set of

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sources. In [6,7], Belnap proposed a propositional framework to this end, based on Dunn's four-valued matrix [11]. In his model, sources of information are only allowed to provide information on *atomic* formulas. However, this model is inadequate for dealing with knowledge bases in which information about complex formulas may not originate from information about atomic formulas. Therefore Belnap's framework is generalized in [2] to the Existential Information Processing (EIP) framework, where sources may provide information on complex formulas too. For example, a source which does not state that φ is true, nor that ψ is true, may still state that their disjunction is true. For reasoning under those circumstances, a corresponding strongly sound and complete Gentzen-type proof system, that admits strong cut-elimination, is provided.

The EIP framework of [2] is still confined to the propositional level. However, a knowledge base should permit queries in a first-order language in order to really be useful. Using our extension to quasi-canonical systems we are able to extend the EIP framework to the first-order level, carrying over its induced semantics and proof system, and prove that the latter admits strong cut-elimination.

1 Preliminaries

The following conventions are used throughout this paper.

- $\mathbb{N}$ is the set of natural numbers (which includes 0).
- A prefix of $\mathbb{N}$ is a set $\{n \in \mathbb{N} \mid n < k\}$, where $k \in \mathbb{N} \cup \{\infty\}$.
- A function $f : X \to Y$ where $X \cap \mathcal{P}[X] = \emptyset$ is implicitly extended to $f : X \cup \mathcal{P}[X] \to Y \cup \mathcal{P}[Y]$ by acting point-wise, i.e.

$$f[\zeta] = \begin{cases} f[\zeta] & \zeta \in X \\ \{f[z] \mid z \in \zeta\} & \zeta \subseteq X \end{cases}$$

This paper considers first-order languages with dual-arity quantifiers, i.e. $\langle n, k\rangle$-quantifiers for some $n, k \in \mathbb{N}$. Such a quantifier connects n formulas and binds k variables. Connectives of arity n are seen as $\langle n, 0\rangle$-quantifiers.

Example 1. The language of first-order logic is usually defined to have the $\langle 1, 0\rangle$-quantifier $\neg$, the $\langle 2, 0\rangle$-quantifiers $\vee$, $\wedge$, $\to$, and the $\langle 1, 1\rangle$-quantifiers $\exists$, $\forall$.

> For the rest of this paper L is a fixed first-order language with dual-arity quantifiers. Constants of L are taken as 0-ary function symbols.

Construction of L-terms and atomic L-formulas is standard, and that of L-formulas is a simple generalization of the usual construction: If Q is an $\langle n, k\rangle$-ary quantifier in L, $z_1, \dots z_k$ are distinct variables, and $A_1, \dots A_n$ are L-formulas, then $\mathsf{Q}\, z_1 \dots z_k\, (A_1, \dots A_n)$ is an L-formula where the free occurrences of $z_1, \dots z_k$ in each of the formulas $A_1, \dots A_n$ become bound. Here Q is said to connect $A_1, \dots A_n$ and bind $z_1, \dots z_k$.

If A and A' are L-formulas that are equal up to renaming bound variables, we write $A \overset{\alpha}{\sim} A'$. If A is an L-formula, $t_1, \dots t_k$ are L-terms, and $z_1, \dots z_k$ are distinct variables, then $A\{t_1/z_1,\dots t_k/z_k\}$ is obtained from A by simultaneously replacing free occurrences of z_i by t_i for all $i \in \{1,\dots k\}$. The accompanying concept of $t_1,\dots t_k$ being substitutable for $z_1,\dots z_k$ in A is defined as usual.

An L-sequent is a construct of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of L-formulas. We make use of the list-for-union shorthand, e.g. $\Gamma, A, \Delta, B \Rightarrow$ stands for $\{A, B\} \cup \Gamma \cup \Delta \Rightarrow \emptyset$.

Definition 1. *Let $V \subseteq \mathsf{Var}$ (the set of all variables). An L-formula (-term) is* **V-open** *if it has no free variables outside of V; it is* **closed** *if it is $\emptyset$-open.*

Non-deterministic matrices [4] provide a rich and modular semantic framework. First defined for propositional logic, the concept was later generalized to predicate logic with dual-arity quantifiers [3]. In what follows, we restrict the domain of our structure to at most countable, so without loss of generality the domain may be taken to be a prefix of $\mathbb{N}$.

Definition 2. *A* **generalized non-deterministic matrix (GNmatrix) for** *L is a triple $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ such that:*

- *$\mathcal{V}$ is a set (of truth values).*
- *$\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$ (of designated truth values).*
- *$\mathcal{O}$ associates with every non-empty prefix X of $\mathbb{N}$ and every $\langle n, k \rangle$-quantifier Q a function $\tilde{\mathsf{Q}}_X : \left(X^k \to \mathcal{V}^n\right) \to \mathcal{P}^+[\mathcal{V}]$ (truth table).*

Note that the quantifiers' interpretations return *sets* of truth values. This will give rise to the semantics' non-determinism, specifically in Definition 12 below.

▌ For the rest of this section $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a fixed GNmatrix.

Notation. If Q is a connective, $\tilde{\mathsf{Q}}_X$ may be abbreviated to $\tilde{\mathsf{Q}}$.

Definition 3. *An* **L-algebra** *(in the sense of [5]) $\mathcal{A}$ consists of:*

- *A non-empty prefix $\mathsf{Dom}\,\mathcal{A}$ of $\mathbb{N}$ called the* **domain of $\mathcal{A}$**.
- *For each m-ary func. symbol f in L, a function $f^{\mathcal{A}} : (\mathsf{Dom}\,\mathcal{A})^m \to \mathsf{Dom}\,\mathcal{A}$ called the* **interpretation of f in $\mathcal{A}$**.

▌ For the rest of this section $\mathcal{A}$ is a fixed L-algebra.

Notation. If t is a closed L-term, then $t^{\mathcal{A}}$ denotes its interpretation in the L-algebra $\mathcal{A}$, defined inductively: $\left(f\left(t_1,\dots t_m\right)\right)^{\mathcal{A}} = f^{\mathcal{A}}\left[{t_1}^{\mathcal{A}},\dots {t_m}^{\mathcal{A}}\right]$.

Definition 4. *An* **$\mathcal{A}$-based L-informer for** *$\mathcal{M}$, $\mathcal{I}$, consists of the following:*

- *For every m-ary predicate symbol p in L, a predicate $p^{\mathcal{I}} : (\mathsf{Dom}\,\mathcal{A})^m \to \mathcal{V}$ called the* **interpretation of p in $\mathcal{I}$**.

▌ For the rest of this section $\mathcal{I}$ is a fixed $\mathcal{A}$-based L-informer for $\mathcal{M}$.

Definition 5. *A pair* $\langle\mathcal{A},\mathcal{I}\rangle$*, where* $\mathcal{A}$ *and* $\mathcal{I}$ *are as above, is called an* **L-structure for** $\mathcal{M}$ *(which is* **based at** $\mathcal{A}$ *and* **informed by** $\mathcal{I}$*).*[1]

▌ For the rest of this section $\mathcal{S} = \langle\mathcal{A},\mathcal{I}\rangle$ is a fixed L-structure for $\mathcal{M}$.

Substitutional semantics [14] is used to handle assignment of elements of the domain to free variables when evaluating a formula. This contrasts with the prevailing denotational semantics which is inadequate in the non-deterministic context. What follows is a condensed and slightly adapted presentation of notions that appear in [4] (for more see references there).

Definition 6. *The set* $\{\overline{a} \mid a \in \mathsf{Dom}\,\mathcal{A}\}$ *of the* **individual constants of** $\mathcal{A}$ *is obtained by associating a constant with every member of* $\mathsf{Dom}\,\mathcal{A}$*.*

Notation. $L(\mathcal{A})$ is obtained by extending L with $\{\overline{a} \mid a \in \mathsf{Dom}\,\mathcal{A}\}$.

Definition 7. *The extension of* $\mathcal{A}$ *to an* $L(\mathcal{A})$*-algebra is obtained by letting* $\overline{a}^{\mathcal{A}} = a$ *for every* $a \in \mathsf{Dom}\,\mathcal{A}$*.*

Definition 8. *An* **$\mathcal{A}$-substitution** *is a* $\mathsf{Var} \to \{\overline{a} \mid a \in \mathsf{Dom}\,\mathcal{A}\}$ *function.*

Definition 9. *Let* t *be an* $L(\mathcal{A})$*-term. The* **normal form of** t*, denoted* $|t|$*, is defined inductively as follows:*

- *If* $t = f(t_1, \ldots t_m)$*, then* $|t| = \overline{t^{\mathcal{A}}}$ *if* t *is closed, otherwise* $|t| = f(|t_1|, \ldots |t_m|)$*.*
- *Otherwise (i.e.* t *is a variable),* $|t| = t$*.*

For an $L(\mathcal{A})$*-term* t'*, we write* $t \overset{\mathcal{A}}{\sim} t'$ *if* $|t| = |t'|$*.*

Definition 10. *Let* φ*,* φ' *be* $L(\mathcal{A})$*-formulas. We write* $\varphi \overset{\mathcal{A}}{\sim} \varphi'$ *if* $|\varphi| \overset{\alpha}{\sim} |\varphi'|$*, where* $|\varphi|$*, the* **normal form of** φ*, is defined inductively as follows:*

- *If* $\varphi = p(t_1, \ldots t_m)$ *is atomic, then* $|\varphi| = p(|t_1|, \ldots |t_m|)$*.*
- *If* $\varphi = \mathsf{Q}\, z_1 \ldots z_k\, (\psi_1, \ldots \psi_n)$*, then* $|\varphi| = \mathsf{Q}\, z_1 \ldots z_k\, (|\psi_1|, \ldots |\psi_n|)$*.*

Valuations are functions that assign truth values to all formulas in a way that is compatible with a particular GNmatrix and structure. In many cases, and in §3 specifically, it is desirable to define valuations only on some of the formulas.

Definition 11. *A set* Φ *of* $L(\mathcal{A})$*-formulas is* **closed under subsentences (Sclosed)** *if every formula in* Φ *is closed, and* $\mathsf{Q}\, z_1 \ldots z_k\, (\psi_1, \ldots \psi_n) \in \Phi$ *implies that for all* $i \in \{1, \ldots n\}$ *and* $a_1, \ldots a_k \in \mathsf{Dom}\,\mathcal{A}$*,* $\psi_i\,\{\overline{a_1}/z_1, \ldots \overline{a_k}/z_k\} \in \Phi$*.*

Notation. $\tilde{\mathsf{Q}}_{\mathsf{Dom}\,\mathcal{A}}$ may be abbreviated to $\tilde{\mathsf{Q}}_{\mathcal{A}}$.

[1] This is equivalent to the usual definition of a structure. However, it is more convenient for our purposes. See e.g. the independence of Definitions 9 and 10 below from the informer, and the statement of Proposition 2. The convenience is further evident in §3, where the base algebra remains fixed while the informer varies.

Definition 12. *Let Φ be an Sclosed set of $L(\mathcal{A})$-formulas, and let $v : \Phi \to \mathcal{V}$. Consider the following conditions:*

A. *If $\varphi \overset{\mathcal{A}}{\sim} \varphi'$, then $v[\varphi] = v[\varphi']$.*
I. $v[p(t_1, \dots t_m)] = p^{\mathcal{I}}\left(t_1{}^{\mathcal{A}}, \dots t_m{}^{\mathcal{A}}\right)$.
Q. $v[\mathsf{Q}\, z_1 \dots z_k (\psi_1, \dots \psi_n)] \in \tilde{\mathsf{Q}}_{\mathcal{A}}[h]$, *where h is* $\lambda a_1, \dots a_k \in \mathsf{Dom}\,\mathcal{A}\,.\,\langle v[\psi_1\{\overline{a_1}/z_1, \dots \overline{a_k}/z_k\}], \dots v[\psi_n\{\overline{a_1}/z_1, \dots \overline{a_k}/z_k\}]\rangle$.

- *v is a* **partial $\mathcal{M}$-legal $\mathcal{A}$-valuation** *if conditions A and Q hold.*
- *v is a* **partial $\mathcal{M}$-legal $\mathcal{S}$-valuation** *if conditions A, I and Q hold.*
- *The word 'partial' may be omitted if Φ includes all closed $L(\mathcal{A})$-formulas.*

Proposition 1. *[4] Every partial $\mathcal{M}$-legal $\mathcal{S}$-valuation v is extendable to an $\mathcal{M}$-legal $\mathcal{S}$-valuation (and similarly for partial $\mathcal{M}$-legal $\mathcal{A}$-valuations).*

Proposition 2. *For every partial $\mathcal{M}$-legal $\mathcal{A}$-valuation v there exists an $\mathcal{A}$-based L-informer $\tilde{\mathcal{I}}$ for $\mathcal{M}$ such that v is a partial $\mathcal{M}$-legal $\left\langle \mathcal{A}, \tilde{\mathcal{I}} \right\rangle$-valuation.*

Definition 13. *Let C be an L-formula, $\Theta \cup \{\Gamma \Rightarrow \Delta\}$ be a set of L-sequents, v be an $\mathcal{M}$-legal $\mathcal{S}$-valuation, and σ be an $\mathcal{S}$-substitution. Define:*

- $\mathcal{S}, v, \sigma \models C$ *if* $v[\sigma[C]] \in \mathcal{D}$.
- $\mathcal{S}, v, \sigma \models \Gamma \Rightarrow \Delta$ *if there exists $A \in \Gamma$ such that $\mathcal{S}, v, \sigma \not\models A$ or $B \in \Delta$ such that $\mathcal{S}, v, \sigma \models B$.*
- $\mathcal{S}, v, \sigma \models \Theta$ *if $\mathcal{S}, v, \sigma \models \Gamma' \Rightarrow \Delta'$ for every $\Gamma' \Rightarrow \Delta' \in \Theta$.*
- $\mathcal{S}, v \models \star$ *if $\mathcal{S}, v, \sigma' \models \star$ for every $\mathcal{S}$-substitution σ' ($\star$ is a formula, sequent, or set of sequents).*
- $\Theta \vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$ *if the following holds for every L-structure $\mathcal{S}'$ for $\mathcal{M}$ and $\mathcal{M}$-legal $\mathcal{S}'$-valuation v': if $\mathcal{S}', v' \models \Theta$, then $\mathcal{S}', v' \models \Gamma \Rightarrow \Delta$.*[2]

Definition 14. *Let $\Theta \cup \{\Gamma \Rightarrow \Delta\}$ be a set of L-sequents.*

- $\Theta \vdash_G \Gamma \Rightarrow \Delta$ *if $\Gamma \Rightarrow \Delta$ is derivable from Θ in G.*
- *G is* **strongly sound** *for $\mathcal{M}$ if $\vdash_G \subseteq \vdash_{\mathcal{M}}$.*
- *G is* **strongly complete** *for $\mathcal{M}$ if $\vdash_{\mathcal{M}} \subseteq \vdash_G$.*
- *$\mathcal{M}$ is* **strongly characteristic** *for G if G is strongly sound and strongly complete for $\mathcal{M}$.*

Notation. It will often be convenient to use a structure instead of its base algebra or informer:

- $\mathsf{Dom}\,\mathcal{S} = \mathsf{Dom}\,\mathcal{A}$; $L(\mathcal{S}) = L(\mathcal{A})$; $\overset{\mathcal{S}}{\sim} = \overset{\mathcal{A}}{\sim}$.
- For a function symbol f: $f^{\mathcal{S}} = f^{\mathcal{A}}$; and for a closed term t: $t^{\mathcal{S}} = t^{\mathcal{A}}$.
- For a predicate symbol p: $p^{\mathcal{S}} = p^{\mathcal{I}}$.

[2] Two consequence relations for formulas $\Gamma \vdash_{\mathcal{M}} \varphi$ are definable using this consequence relation for sequents: 'truth' $\vdash_{\mathcal{M}} \Gamma \Rightarrow \varphi$ and 'validity' $\{\Rightarrow \psi \mid \psi \in \Gamma\} \vdash_{\mathcal{M}} \Rightarrow \varphi$.

2 Quasi-canonical Proof Systems and Their Semantics

As we said in the introduction, a characterization for strong cut-elimination was given in [3] for *canonical* Gentzen-type systems. Specifically, the following properties of a canonical system G with dual-arity quantifiers are shown to be equivalent: (i) G is coherent, (ii) G admits strong cut-elimination, and (iii) G has a strongly characteristic GNmatix of a particular kind. In [1] it is shown that for a *quasi*-canonical system G of *proposition* logic (i) entails (ii) and (iii). This section combines these two results, thus generalizing both (Theorem 1): (i), (ii) and (iii) are found to be equivalent for a quasi-canonical system G with dual-arity quantifiers.

▌ For the rest of this paper assume L includes the 1-ary connective $\neg$.

2.1 Introducing Quasi-canonical Proof Systems

The family of highly simplified representation languages defined below suffices for expressing the logical rules of a quasi-canonical system.

Definition 15. *The language L_k^n is the language that consists – aside from the mandatory variables and auxiliary symbols – of enumerably many constants $\mathsf{Con} = \{c_i \mid i \in \mathbb{N}\}$, predicate symbols $p_1, \dots p_n$ of arity k, and the connective $\neg$.*

Notation. Let $\overset{a}{\neg}$ denote $\neg$ if $a = 1$, and the empty string if $a = 0$.

Definition 16. *An $\langle n, k\rangle$-**literal** is an L_k^n-formula of the form $\overset{a}{\neg} p_i\,(t_1, \dots t_k)$, where $a \in \{0, 1\}$, $i \in \{1, \dots n\}$, and for every $j \in \{1, \dots k\}$, $t_j \in \mathsf{Con} \cup \mathsf{Var}$. An $\langle n, k\rangle$-**gc** (generalized clause) is a sequent of $\langle n, k\rangle$-literals.*

Definition 17. *Let Q be an $\langle n, k\rangle$-ary quantifier. A **quasi-canonical rule for** Q is a construct of the form Λ/T, where Λ is a set of $\langle n, k\rangle$-gcs, and T is the rule's type — one of the following: $(\mathsf{Q} \Rightarrow)$, $(\Rightarrow \mathsf{Q})$, $(\neg\,\mathsf{Q} \Rightarrow)$, $(\Rightarrow \neg\,\mathsf{Q})$. An $\langle n, k\rangle$-**rule** is a quasi-canoncial rule for an $\langle n, k\rangle$-ary quantifier.*

To apply an $\langle n, k\rangle$-rule as an inference in a proof one must first instantiate the schematic constituents of L_k^n by constituents of L.

Definition 18. *Let $r = \Lambda/T$ be an $\langle n, k\rangle$-rule. Let Φ be a set of L-formulas and $z_1, \dots z_k$ be distinct variables. An $\langle L, r, \Phi, z_1, \dots z_k\rangle$-**mapping** is any function χ from the terms and predicate symbols of L_k^n to terms and formulas of L, satisfying the following conditions:*

- *For every $y \in \mathsf{Var}$, $\chi\,[y] \in \mathsf{Var}$, and for every $x \in \mathsf{Var}$ such that $x \neq y$, $\chi\,[x] \neq \chi\,[y]$.*
- *For every $c \in \mathsf{Con}$, $\chi\,[c]$ is an L-term, such that for every $x \in \mathsf{Var}$ occurring in Λ, $\chi\,[x]$ does not occur in $\chi\,[c]$.*

– *For every* $i \in \{1, \dots n\}$, $\chi[p_i]$ *is an* L*-formula. If* $\overset{a}{\neg} p_i(t_1, \dots t_k)$ *occurs in* Λ, *then for every* $j \in \{1, \dots k\}$: $\chi[t_j]$ *is substitutable for* z_j *in* $\chi[p_i]$, *and if* $t_j \in \mathsf{Var}$, *then* $\chi[t_j]$ *does not occur free in* $\Phi \cup \{\mathsf{Q}\, z_1 \dots z_k (\chi[p_1], \dots \chi[p_n])\}$.

χ *extends to* $\langle n, k\rangle$*-literals by* $\chi\left[\overset{a}{\neg} p_i(t_1, \dots t_n)\right] = \overset{a}{\neg}\chi[p_i]\{\chi[t_1]/z_1, \dots \chi[t_k]/z_k\}$.

Definition 19. *Let* Q *be an* $\langle n, k\rangle$*-ary quantifier, and* $r = \{\Pi_\ell \Rightarrow \Sigma_\ell\}_{\ell=1}^m / (\mathsf{Q} \Rightarrow)$ *be a quasi-canonical rule for* Q. *An* **application of** r *is any inference step of the form:*

$$\frac{\{\Gamma, \chi[\Pi_\ell] \Rightarrow \chi[\Sigma_\ell], \Delta\}_{\ell=1}^m}{\Gamma, \mathsf{Q}\, z_1 \dots z_k (\chi[p_i], \dots \chi[p_n]) \Rightarrow \Delta}\ (\mathsf{Q} \Rightarrow)$$

where χ *is some* $\langle L, r, \Gamma \cup \Delta, z_1, \dots z_k\rangle$*-mapping.*

Applications of the other types of quasi-canonical rules are defined similarly.

Example 2. Consider the following quasi-canonical rules for $\exists$:

$$\{\Rightarrow \neg p_1(v_1)\} / (\Rightarrow \neg\exists) \qquad \{\neg p_1(c_1) \Rightarrow\} / (\neg\exists \Rightarrow)$$

Application of these rules has the forms:

$$\frac{\Gamma \Rightarrow \neg A\{x/z\}, \Delta}{\Gamma \Rightarrow \neg\exists z A, \Delta}\ (\Rightarrow \neg\exists) \qquad \frac{\Gamma, \neg A\{t/z\} \Rightarrow \Delta}{\Gamma, \neg\exists z A \Rightarrow \Delta}\ (\neg\exists \Rightarrow)$$

where x is not free in $\Gamma \cup \Delta \cup \{\neg\exists z A\}$, and x and t are substitutable for z in A.

Definition 20. *A* **full quasi-canonical calculus for** L *is a Gentzen-type system that consists of rules of the following types:*

– *Logical rules: a finite number of quasi-canonical inference rules.*
– *Structural rules: the* α*-axiom scheme* (A), *the weakening rule* (W), *the cut rule* (C), *and the substitution rule* (S), *with application forms*

$$\frac{}{A \Rightarrow A'}\ (\mathsf{A}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\ (\mathsf{W}) \qquad \frac{\Gamma' \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta'}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\ (\mathsf{C})$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma\{t_1/x_1, \dots t_m/x_m\} \Rightarrow \Delta\{t_1/x_1, \dots t_m/x_m\}}\ (\mathsf{S})$$

where $\Gamma, \Gamma', \Delta, \Delta', \{A, A'\}$ *are sets of* L*-formulas such that* $A \overset{\alpha}{\sim} A'$; $x_1, \dots x_m$ *are distinct variables;* $t_1, \dots t_m$ *are* L*-terms substitutable for* $x_1, \dots x_m$ *in every formula in* $\Gamma \cup \Delta$.

A **full 4-quasi-canonical calculus for** L *is a full quasi-canonical calculus in which there are no rules of the types* $(\neg \Rightarrow)$ *and* $(\Rightarrow \neg)$.

The structural rules are sound in the following sense:

Proposition 3. *Let $\Theta \cup \{\Gamma \Rightarrow \Delta\}$ be a set of L-sequents such that $\Theta / \Gamma \Rightarrow \Delta$ is an application of a structural rule r. Let $\mathcal{M}$ be an GNmatix. Let $\mathcal{S}$ be an L-structure for $\mathcal{M}$, and v be a $\mathcal{M}$-legal $\mathcal{S}$-valuation, such that $\mathcal{S}, v \models \Theta$. Let σ be an $\mathcal{S}$-substitution. Then $\mathcal{S}, v, \sigma \models \Gamma \Rightarrow \Delta$.*

Proof. By case analysis on the rule r:

- If r is the α-axiom (A) this follows from the definition of valuations (and $\overset{\mathcal{S}}{\sim}$).
- If r is the weakening rule (W) or cut rule (C) this follows trivially as usual.
- If r is the substitution rule (S), then for every variable x, denote by t_x its L-term replacement (or simply $t_x = x$ if x was not replaced). Let σ' be the $\mathcal{S}$-substitution such that $\sigma' [x] = \overline{(\sigma [t_x])}^{\mathcal{S}}$. In particular, $\sigma' [x] \overset{\mathcal{S}}{\sim} \sigma [t_x]$. By assumption, $\mathcal{S}, v, \sigma' \models \Theta$. Consequently, $\mathcal{S}, v, \sigma \models \Gamma \Rightarrow \Delta$.

Coherence [3] is a syntactic property of quasi-canonical systems that will later be used to determine whether the system admits strong cut-elimination.

Definition 21. *A set Λ of $\langle n, k\rangle$-gcs is* **inconsistent** *if there is a proof of $\Rightarrow$ from Λ using only* (C) *and* (S)*; otherwise it is* **consistent**.

Definition 22. *Let Λ_1 and Λ_2 be sets of $\langle n, k\rangle$-gcs. $\Lambda_1 \Cup \Lambda_2$ is $\Lambda_1 \cup \Lambda_2'$, where Λ_2' is obtained from Λ_2 by fresh renaming of constants and variables in Λ_1.*

Definition 23. *Rules Λ_1/T_1 and Λ_2/T_2 are* **conflicting** *if for some quantifier* Q *either $T_1 = (\mathsf{Q} \Rightarrow)$ and $T_2 = (\Rightarrow \mathsf{Q})$, or $T_1 = (\neg\, \mathsf{Q} \Rightarrow)$ and $T_2 = (\Rightarrow \neg\, \mathsf{Q})$.*

Definition 24. *A full quasi-canonical calculus for L is* **coherent** *if for every pair of conflicting rules Λ_1/T_1 and Λ_2/T_2, the set $\Lambda_1 \Cup \Lambda_2$ is inconsistent.*

Example 3. Consider the full quasi-canonical calculus in which the inference rules are those from Example 2. These rules are conflicting. However, the set $\{\neg p_1 (c_1) \Rightarrow, \Rightarrow \neg p_1 (v_1)\}$ is clearly inconsistent, so the calculus is coherent.

Proposition 4. *Let $\Lambda \cup \{\Pi \Rightarrow \Sigma\}$ be a set of $\langle n, k\rangle$-gcs. If there is a proof of $\Pi \Rightarrow \Sigma$ from Λ using only* (C) *and* (S)*, then there is such a proof in which* (S) *is used only as the first inference step on leaves of the proof tree, and only for substituting by constants that appear in $\Lambda \cup \{\Pi \Rightarrow \Sigma\}$.*

Proof. Note that an application of (C) followed by an application of (S) can be replaced with an a pair of applications of (S) followed by an application of (C); and two consecutive applications of (S) can be replaced with one. Using induction on the given proof's height, applications of (S) can thus be pushed to the leaves. Next, using induction on the given proof's height, the obtained proof remains valid after replacing all variables and constants that do not appear in $\Lambda \cup \{\Pi \Rightarrow \Sigma\}$ with a variable or constant that does appear in $\Lambda \cup \{\Pi \Rightarrow \Sigma\}$.

Corollary 1. *The coherence of a full 4-quasi-canonical calculus is decidable.*

2.2 The Semantics of Quasi-canonical Proof Systems

The semantics of quasi-canonical proof systems is based on Dunn's four truth values [11,13], where each truth value is a different subset of $\{0,1\}$, and the presence of 1 (0) indicates evidence supporting (opposing) the truth of a formula.

Notation. $\bot = \{\}$; $\mathrm{f} = \{0\}$; $\mathrm{t} = \{1\}$; $\top = \{0,1\}$.

A statement is considered true iff it has supporting evidence, and its negation true iff the statement has opposing evidence.

Definition 25. *A GNmatix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for L is a* **$\neg$-GNmatix** *if:*

- $\mathcal{V} \subseteq \{\mathrm{t}, \mathrm{f}, \top, \bot\}$, *and* $\mathcal{D} = \mathcal{V} \cap \{\mathrm{t}, \top\}$.
- *The following hold for the operation $\tilde{\neg}$ of $\mathcal{O}$:*
  - *If* $\mathrm{t} \in \mathcal{V}$, *then* $\tilde{\neg}\mathrm{t} \subseteq \{\mathrm{f}, \bot\}$.
  - *If* $\top \in \mathcal{V}$, *then* $\tilde{\neg}\top \subseteq \{\mathrm{t}, \top\}$.
  - *If* $\mathrm{f} \in \mathcal{V}$, *then* $\tilde{\neg}\mathrm{f} \subseteq \{\mathrm{t}, \top\}$.
  - *If* $\bot \in \mathcal{V}$, *then* $\tilde{\neg}\bot \subseteq \{\mathrm{f}, \bot\}$.
- *All operations $\tilde{\mathsf{Q}}$ of $\mathcal{O}$ return members of*

$$\{\mathcal{V}, \{\mathrm{t}, \top\}, \{\mathrm{t}, \bot\}, \{\mathrm{f}, \top\}, \{\mathrm{f}, \bot\}, \{\mathrm{t}\}, \{\mathrm{f}\}, \{\top\}, \{\bot\}\}$$

Definition 26. $\mathcal{M}_4 = \left\langle \{\mathrm{t}, \mathrm{f}, \top, \bot\}, \{\mathrm{t}, \top\}, \left\{\tilde{\neg}^4\right\} \right\rangle$ *with* $\tilde{\neg}^4\mathrm{t} = \tilde{\neg}^4\bot = \{\mathrm{f}, \bot\}$, $\tilde{\neg}^4\mathrm{f} = \tilde{\neg}^4\top = \{\mathrm{t}, \top\}$.

The next couple of definitions are adapted from [1]. First, a function is defined to take a quasi-canonical rule for some quantifier Q and return a set of truth values. Intuitively, the set returned consists of those truth values Q can take for the rule's conclusion to hold.

Definition 27. *The function F on quasi-canonical rules is defined as follows:*

$$F[r] = \begin{cases} \{\mathrm{t}, \top\} & r \text{ is of type } (\Rightarrow \mathsf{Q}) \\ \{\mathrm{f}, \bot\} & r \text{ is of type } (\mathsf{Q} \Rightarrow) \\ \{\mathrm{f}, \top\} & r \text{ is of type } (\Rightarrow \neg\, \mathsf{Q}) \\ \{\mathrm{t}, \bot\} & r \text{ is of type } (\neg\, \mathsf{Q} \Rightarrow) \end{cases}$$

Next, the function is used to provide an interpretation to quantifiers that corresponds to a given Gentzen-type proof system.

Definition 28. *Let G be a coherent full 4-quasi-canonical calculus for L. The* **$\neg$-GNmatrix induced by** *G, denoted $\mathcal{M}_G$, is the $\neg$-GNmatrix $\langle \mathcal{V}_4, \{\mathrm{t}, \top\}, \mathcal{O}_G \rangle$ in which, for every non-empty prefix X of $\mathbb{N}$, the interpretation $\tilde{\mathsf{Q}}_X$ in $\mathcal{O}_G$ of an $\langle n, k \rangle$-quantifier Q in L is defined as follows:*

$$\tilde{\mathsf{Q}}_X[h] = \begin{cases} \bigcap \{F[r] \mid r \in R_G[\mathsf{Q}, X, h]\} & \mathsf{Q} \neq \neg \\ \tilde{\neg}^4[h] \cap \bigcap \{F[r] \mid r \in R_G[\mathsf{Q}, X, h]\} & \mathsf{Q} = \neg \end{cases}$$

where $R_G[\mathsf{Q}, X, h]$ is the set of rules Λ/T for Q in G that satisfy the following: an L^n_k-structure $\mathcal{N}$ for $\mathcal{M}_4$ exists such that $\mathsf{Dom}\,\mathcal{N} = X$, ${p_i}^{\mathcal{N}} = h_i$, and $\mathcal{N} \models \Lambda$.

Examples where Definitions 27 and 28 are employed can be found in the proof of Theorem 2 below.

Proposition 5. *Let G be a coherent full 4-quasi-canonical calculus for L. Then $\mathcal{M}_G$ is a well-defined four-valued $\neg$-GNmatix.*

2.3 Soundness, Completeness, and Cut-elimination

Proposition 6. *Let G be a coherent full 4-quasi-canonical calculus for L. Then G is strongly sound for $\mathcal{M}_G$.*

Definition 29. *Let G be a full quasi-canonical calculus.*

- *Let $\Theta \cup \{\Gamma \Rightarrow \Delta\}$ be some set of L-sequents. A proof in G of $\Gamma \Rightarrow \Delta$ from Θ is* Θ**-cut-free** *if all cuts in the proof are on substitution instances of formulas from Θ.*
- *G admits* **strong cut-elimination** *if for every set of L-sequents $\Theta \cup \{\Gamma \Rightarrow \Delta\}$ that satisfies the free-variable condition (no variable occurs both free and bound): if there is a proof in G of $\Gamma \Rightarrow \Delta$ from Θ, there is also such a proof which is Θ-cut-free.*

Example 4. Consider the following proofs of $\Rightarrow$ from $\{\Rightarrow \neg p(x), \neg p(c) \Rightarrow\}$ in the system from Example 3:

$$\dfrac{\dfrac{\Rightarrow \neg p(x)}{\Rightarrow \neg\exists x p(x)}\,(\Rightarrow\neg\exists) \qquad \dfrac{\neg p(c) \Rightarrow}{\neg\exists x p(x) \Rightarrow}\,(\neg\exists\Rightarrow)}{\Rightarrow}\,(\mathsf{C}) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Rightarrow \neg p(x)}{\Rightarrow \neg p(c)}\,(\mathsf{S}) \qquad \neg p(c) \Rightarrow}{\Rightarrow}\,(\mathsf{C})$$

The cut in the proof on the left was eliminated by using the substitution rule, resulting in the proof on the right which is $\{\Rightarrow \neg p(x), \neg p(c) \Rightarrow\}$-cut-free.

Proposition 7. *Let G be a coherent full 4-quasi-canonical calculus. Let $\Theta \cup \{\Gamma \Rightarrow \Delta\}$ be a set of L-sequents that satisfies the free-variable condition. If $\Gamma \Rightarrow \Delta$ has no Θ-cut-free proof from Θ in G, then $\Theta \nvdash_{\mathcal{M}_G} \Gamma \Rightarrow \Delta$.*

Proposition 8. *Let $\Lambda \cup \{\Pi \Rightarrow \Sigma\}$ be a set of $\langle n, k\rangle$-gcs.*

1. *If there is a proof of $\Pi \Rightarrow \Sigma$ from Λ using only* (A), (W), (C), *and* (S), *then there are $\Pi' \subseteq \Pi$ and $\Sigma' \subseteq \Sigma$ such that there is a proof of $\Pi' \Rightarrow \Sigma'$ from Λ using only* (A), (C), *and* (S).
2. *If there is a proof of $\Pi \Rightarrow \Sigma$ from Λ using only* (A), (C), *and* (S), *and $\Pi \Rightarrow \Sigma$ is not an instance of* (A), *then there is a proof of $\Pi \Rightarrow \Sigma$ from Λ using only* (C) *and* (S).

Corollary 2. *If a set Λ of $\langle n, k\rangle$-gcs is consistent, then there is an L^n_k-structure $\mathcal{N}$ for $\mathcal{M}_4$ such that $\mathcal{N} \models \Lambda$.*

Theorem 1. *Let G be a full 4-quasi-canonical calculus for L. The following are equivalent:*

1. *G is coherent.*
2. *G is coherent and $\mathcal{M}_G$ is strongly characteristic for G.*
3. *G has a strongly characteristic $\neg$-GNmatix.*
4. *G admits strong cut-elimination.*

Proof. We prove $1 \implies 2 \implies 3 \implies 1$ and $1 \implies 4 \implies 1$:

1 $\Longrightarrow$ 2. Assume G is coherent. Then by Proposition 6, $\mathcal{M}_G$ is strongly sound for G. It remains to show that $\mathcal{M}_G$ is strongly complete for G. Let $\Theta \cup \{\Gamma \Rightarrow \Delta\}$ be a set of L-sequents such that $\Gamma \Rightarrow \Delta$ has no proof from Θ in G. Rename variables in $\Theta \cup \{\Gamma \Rightarrow \Delta\}$ as necessary to obtain $\Theta' \cup \{\Gamma' \Rightarrow \Delta'\}$ satisfying the free-variable condition. Then $\Gamma' \Rightarrow \Delta'$ has no proof from Θ' in G, otherwise a proof of $\Gamma \Rightarrow \Delta$ from Θ in G could be obtained by using (A) and (C). By Proposition 7, $\Theta' \nvdash_{\mathcal{M}_G} \Gamma' \Rightarrow \Delta'$. Since valuations respect α-equivalence, $\Theta \nvdash_{\mathcal{M}_G} \Gamma \Rightarrow \Delta$. Therefore, if $\Theta \vdash_{\mathcal{M}_G} \Gamma \Rightarrow \Delta$, then $\Theta \vdash_G \Gamma \Rightarrow \Delta$, the required strong completeness.

2 $\Longrightarrow$ 3. $\mathcal{M}_G$ is a $\neg$-GNmatix by Proposition 5.

3 $\Longrightarrow$ 1. Assume G has a strongly characteristic $\neg$-GNmatix $\mathcal{M}$. Suppose for contradiction that G is not coherent. Then there must exist two $\langle n, k\rangle$-rules $r_1 = \Lambda_1/\left(\overset{a}{\neg}\,\mathsf{Q} \Rightarrow\right)$ and $r_2 = \Lambda_2/\left(\Rightarrow \overset{a}{\neg}\,\mathsf{Q}\right)$ in G such that $\Lambda_1 \Cup \Lambda_2$ is consistent. By Corollary 2, there exist an L_k^n-structure $\mathcal{N}$ for $\mathcal{M}_4$ and an $\mathcal{M}_4$-legal $\mathcal{N}$-valuation u such that $\mathcal{N}, u \models \Lambda_1 \Cup \Lambda_2$. Pick an L-structure $\mathcal{S}$ that extends[3] $\mathcal{N}$ and an $\mathcal{M}$-legal $\mathcal{S}$-valuation v such that for every closed $L(\mathcal{S})$-literal l it holds that $v[l] \in \{\mathrm{t}, \top\}$ iff $u[l] \in \{\mathrm{t}, \top\}$. Such v exists since $\mathcal{M}$ is a $\neg$-GNmatix. Thus $\mathcal{S}, v \models \Lambda_1 \Cup \Lambda_2$. However, $\Lambda_1 \Cup \Lambda_2 \vdash_G \Rightarrow$, so by strong soundness $\mathcal{S}, v \models \Rightarrow$ which is impossible.

1 $\Longrightarrow$ 4. Let $\Theta \cup \{\Gamma \Rightarrow \Delta\}$ be a set of L-sequents that satisfies the free-variable condition such that $\Theta \vdash_G \Gamma \Rightarrow \Delta$. We have already shown that $\mathcal{M}_G$ is strongly sound for G, and therefore $\Theta \vdash_{\mathcal{M}_G} \Gamma \Rightarrow \Delta$. By Proposition 7, $\Gamma \Rightarrow \Delta$ has a Θ-cut-free proof from Θ in G. Thus G admits strong cut-elimination.

4 $\Longrightarrow$ 1. Assume that G admits strong cut-elimination. Suppose G is not coherent. Then there exist two rules $\Lambda_1/\left(\overset{a}{\neg}\,\mathsf{Q} \Rightarrow\right)$ and $\Lambda_2/\left(\Rightarrow \overset{a}{\neg}\,\mathsf{Q}\right)$ in G such that $\Lambda_1 \Cup \Lambda_2$ is consistent. Obtain $\Lambda_1 \Cup \Lambda_2 \vdash_G \Rightarrow$ by applying each rule once and following with an application of (C). The set $(\Lambda_1 \Cup \Lambda_2) \cup \{\Rightarrow\}$ clearly satisfies the free-variable condition as there are no bound variable occurrences there at all. Since G admits strong cut-elimination, there must be a $\Lambda_1 \Cup \Lambda_2$-cut-free proof in G of $\Rightarrow$ from $\Lambda_1 \Cup \Lambda_2$.

Suppose there was an application of a logical rule in the proof. Since the rule is neither of type $(\neg \Rightarrow)$ nor of type $(\Rightarrow \neg)$, such an application must introduce a non-literal formula. It is easy to show that the existence of a non-literal formula must be retained throughout a proof in which applications of (C) eliminate only literals, in contradiction to the conclusion being $\Rightarrow$.

Therefore, the only rules applied in the proof are (A), (W), (C), and (S). By Proposition 8, the proof can be reduced to one using only (C) and (S). Yet this is a contradiction to the fact that $\Lambda_1 \Cup \Lambda_2$ is consistent.

[3] Without loss of generality, $L_k^n \subseteq L$.

3 Existential Information Processing

In [2] a propositional framework of Existential Information Processing (EIP) is suggested as a means to handle inconsistent information in knowledge bases.[4] This involves indiscriminately gathering information from a set of sources and then processing it in order to discern further logical conclusions, while keeping inconsistencies to a minimum. In this section the framework is extended to predicate logic using the tools developed above.

> For the rest of this paper assume the quantifiers of L are the 1-ary connective $\neg$, the 2-ary connectives $\vee$ and $\wedge$, and the $\langle 1,1\rangle$-ary quantifiers $\exists$ and $\forall$; and assume $\mathcal{A}$ is a fixed L-algebra.

3.1 Sources of Information

In the EIP framework, sources provide information on *arbitrary* formulas, in the form of truth values from $\{\mathrm{i}, 0, 1\}$, where i means that the source doesn't know. This fact enables them to possess disjunctive information: a source may know that $\varphi \vee \psi$ holds without knowing which of φ and ψ holds; and dually, a source may know that $\varphi \wedge \psi$ does not hold without knowing which of φ and ψ does not hold. To extend this framework to predicate logic, sources must provide information on formulas with the classical quantifiers. This will be done here by following the classical intuition that $\exists x\varphi \equiv \bigvee_a \varphi\{\overline{a}/x\}$ and $\forall x\varphi \equiv \bigwedge_a \varphi\{\overline{a}/x\}$, where a ranges over the domain (which may be infinite).

Definition 30. *Let* $\mathcal{QM}^3_r = \left\langle \{\mathrm{i}, 0, 1\}, \{1\}, \mathcal{QO}^3_r \right\rangle$*, where* $\mathcal{QO}^3_r$ *is detailed below:*

a	$\tilde{\neg}a$
i	$\{\mathrm{i}\}$
0	$\{1\}$
1	$\{0\}$

$\tilde{\vee}$	i	0	1
i	$\{\mathrm{i},1\}$	$\{\mathrm{i}\}$	$\{1\}$
0	$\{\mathrm{i}\}$	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$

$\tilde{\wedge}$	i	0	1
i	$\{\mathrm{i},0\}$	$\{0\}$	$\{\mathrm{i}\}$
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{\mathrm{i}\}$	$\{0\}$	$\{1\}$

$h[X]$	$\tilde{\exists}_X[h]$
$\{\mathrm{i}\}$	$\{\mathrm{i},1\}$
$\{\mathrm{i},0\}$	$\{\mathrm{i},1\}$
$\{0\}$	$\{0\}$
else	$\{1\}$

$h[X]$	$\tilde{\forall}_X[h]$
$\{\mathrm{i}\}$	$\{\mathrm{i},0\}$
$\{\mathrm{i},1\}$	$\{\mathrm{i},0\}$
$\{1\}$	$\{1\}$
else	$\{0\}$

Definition 31. *An* $\mathcal{A}$**-source** *is a partial* $\mathcal{QM}^3_r$*-legal* $\mathcal{A}$*-valuation.*
An $\mathcal{A}$**-reservoir** *is a set of* $\mathcal{A}$*-sources.*[5]

Sources in a reservoir share an algebra, thus agreeing on the objects under discussion. This means that disagreement is limited to *properties* of said objects.

> For the rest of this section R is a fixed $\mathcal{A}$-reservoir.

[4] See [8] for a different approach that uses logics of formal inconsistency.
[5] Note how dividing structures into an algebra and an informer is convenient here.

3.2 Gathering and Processing the Information

The next step is to gather the information from the reservoir for processing.

Definition 32. *The* **existential gathering function of** R *is the function* g_R *from the closed* $L(\mathcal{A})$*-formulas to* $\mathcal{V}_4$ *defined as follows:*

$$g_R = \lambda\varphi\,.\,\{b \in \{0,1\} \mid \exists u \in R\,.\, b \in u[\varphi]\}$$

There may be knowledge that can only be learned by processing the information in the reservoir. For example, if source a says φ holds and source b says ψ holds, then the reservoir $\{a, b\}$ provides evidence supporting $\varphi \wedge \psi$. The gatherer will not observe this fact if neither a nor b say $\varphi \wedge \psi$ holds.

Definition 33. *Let* g *be a function from the closed* $L(\mathcal{A})$*-formulas to* $\mathcal{V}_4$*. The* **information processing valuation induced by** g *is the function* d *from the closed* $L(\mathcal{A})$*-formulas to* $\mathcal{V}_4$ *inductively defined as follows (for any* $b \in \{0,1\}$*,* $x \in \mathsf{Var}$*,* θ *an* $\{x\}$*-open* $L(\mathcal{A})$*-formulas, and* $\varphi, \varphi', \varphi_l, \varphi_r$ *closed* $L(\mathcal{A})$*-formulas such that* $\varphi \overset{\mathcal{A}}{\sim} \varphi'$*):*

(d0) $b \in g[\varphi'] \implies b \in d[\varphi]$.
(d1) $b \in d[\varphi] \implies 1 - b \in d[\neg\varphi]$.
(d2) $1 \in d[\varphi_l] \cup d[\varphi_r] \implies 1 \in d[\varphi_l \vee \varphi_r]$.
(d3) $0 \in d[\varphi_l] \cap d[\varphi_r] \implies 0 \in d[\varphi_l \vee \varphi_r]$.
(d4) $1 \in \bigcup_{a \in \mathsf{Dom}\,\mathcal{A}} d[\theta\{\overline{a}/x\}] \implies 1 \in d[\exists x\theta]$.
(d5) $0 \in \bigcap_{a \in \mathsf{Dom}\,\mathcal{A}} d[\theta\{\overline{a}/x\}] \implies 0 \in d[\exists x\theta]$.

The dual items for $\wedge$ *and* $\forall$ *are omitted.*

Proposition 9. *Let* θ, φ *be closed* $L(\mathcal{A})$*-formulas. If* $\theta \overset{\mathcal{A}}{\sim} \varphi$*, then* $d[\theta] = d[\varphi]$*.*

Definition 34. *The* **existential information processing valuation induced by** R*,* d_R*, is the information processing valuation induced by* g_R*.*

Proposition 10. *For existential information processing, (d1), (d3) and (d5) hold in the other direction (* $\Longleftarrow$ *) as well (likewise for their duals).*

These facts permit capturing the semantics of processors using a $\neg$-GNmatix.

Definition 35. *Let* $\mathcal{QM}_E^4 = \langle \mathcal{V}_4, \{\mathrm{t}, \top\}, \mathcal{QO}_E^4 \rangle$*, where* $\mathcal{QO}_E^4$ *is detailed below:*

a	$\tilde{\neg}a$
$\bot$	$\{\bot\}$
f	$\{\mathrm{t}\}$
t	$\{\mathrm{f}\}$
$\top$	$\{\top\}$

$\tilde{\vee}$	$\bot$	f	t	$\top$
$\bot$	$\{\bot,\mathrm{t}\}$	$\{\bot,\mathrm{t}\}$	$\{\mathrm{t}\}$	$\{\mathrm{t}\}$
f	$\{\bot,\mathrm{t}\}$	$\{\mathrm{f},\top\}$	$\{\mathrm{t}\}$	$\{\top\}$
t	$\{\mathrm{t}\}$	$\{\mathrm{t}\}$	$\{\mathrm{t}\}$	$\{\mathrm{t}\}$
$\top$	$\{\mathrm{t}\}$	$\{\top\}$	$\{\mathrm{t}\}$	$\{\top\}$

$\tilde{\wedge}$	$\bot$	f	t	$\top$
$\bot$	$\{\bot,\mathrm{f}\}$	$\{\mathrm{f}\}$	$\{\bot,\mathrm{f}\}$	$\{\mathrm{f}\}$
f	$\{\mathrm{f}\}$	$\{\mathrm{f}\}$	$\{\mathrm{f}\}$	$\{\mathrm{f}\}$
t	$\{\bot,\mathrm{f}\}$	$\{\mathrm{f}\}$	$\{\mathrm{t},\top\}$	$\{\top\}$
$\top$	$\{\mathrm{f}\}$	$\{\mathrm{f}\}$	$\{\top\}$	$\{\top\}$

$h[X]$	$\tilde{\exists}_X[h]$
$\{\bot\}$	$\{\bot,\mathrm{t}\}$
$\{\bot,\mathrm{f}\}$	$\{\bot,\mathrm{t}\}$
$\{\mathrm{f}\}$	$\{\mathrm{f},\top\}$
$\{\mathrm{f},\top\}$	$\{\top\}$
$\{\top\}$	$\{\top\}$
else	$\{\mathrm{t}\}$

$h[X]$	$\tilde{\forall}_X[h]$
$\{\bot\}$	$\{\bot,\mathrm{f}\}$
$\{\bot,\mathrm{t}\}$	$\{\bot,\mathrm{f}\}$
$\{\mathrm{t}\}$	$\{\mathrm{t},\top\}$
$\{\mathrm{t},\top\}$	$\{\top\}$
$\{\top\}$	$\{\top\}$
else	$\{\mathrm{f}\}$

Corollary 3. d_R *is a* $\mathcal{QM}_E^4$*-legal* $\mathcal{A}$*-valuation.*

Proposition 11. *For every* $\mathcal{QM}_E^4$*-legal* $\mathcal{A}$*-valuation* v *there is an* $\mathcal{A}$*-reservoir* R_v *such that* $v = d_{R_v}$*.*

Corollary 4. *The set of all* $\mathcal{QM}_E^4$*-legal* $\mathcal{A}$*-valuations is identical to the set of all existential information processing valuations induced by* $\mathcal{A}$*-reservoirs.*

3.3 Proof System for the Logic Induced by Processors

A Gentzen-type system for processors is defined based on the propositional one from [2] using the same intuition for the quantifiers that was used for $\mathcal{QM}_E^4$.

Definition 36. $\mathbf{QG^4_{EIP}}$ *is the full 4-quasi-canonical calculus for L with the following logical rules:*

$\neg$. $\{\neg\neg p_1 \Rightarrow\} / (\neg\neg \Rightarrow)$, $\{\Rightarrow \neg\neg p_1\} / (\Rightarrow \neg\neg)$.
$\vee$. $\{\Rightarrow p_1, p_2\} / (\Rightarrow \vee)$, $\{\neg p_1, \neg p_2 \Rightarrow\} / (\neg\vee \Rightarrow)$, $\{\Rightarrow \neg p_1, \Rightarrow \neg p_2\} / (\Rightarrow \neg\vee)$.
$\wedge$. $\{p_1, p_2 \Rightarrow\} / (\wedge \Rightarrow)$, $\{\Rightarrow p_1, \Rightarrow p_2\} / (\Rightarrow \wedge)$, $\{\Rightarrow \neg p_1, \neg p_2\} / (\Rightarrow \neg\wedge)$.
$\exists$. $\{\Rightarrow p_1(c_1)\} / (\Rightarrow \exists)$, $\{\neg p_1(c_1) \Rightarrow\} / (\neg\exists \Rightarrow)$, $\{\Rightarrow \neg p_1(v_1)\} / (\Rightarrow \neg\exists)$.
$\forall$. $\{p_1(c_1) \Rightarrow\} / (\forall \Rightarrow)$, $\{\Rightarrow p_1(v_1)\} / (\Rightarrow \forall)$, $\{\Rightarrow \neg p_1(c_1)\} / (\Rightarrow \neg\forall)$.

Figure 1 below presents the application forms of the logical rules of $\mathbf{QG^4_{EIP}}$, where the usual restrictions on variables apply.

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}\ (\neg\neg \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \neg\neg\varphi, \Delta}\ (\Rightarrow \neg\neg)$$

$$\frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ (\Rightarrow \vee) \qquad \frac{\Gamma \Rightarrow \varphi\{t/x\}, \Delta}{\Gamma \Rightarrow \exists x\varphi, \Delta}\ (\Rightarrow \exists)$$

$$\frac{\Gamma, \neg\varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}\ (\neg\vee \Rightarrow) \qquad \frac{\Gamma, \neg\varphi\{t/x\} \Rightarrow \Delta}{\Gamma, \neg\exists x\varphi \Rightarrow \Delta}\ (\neg\exists \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \neg\varphi, \Delta \qquad \Gamma \Rightarrow \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \vee \psi), \Delta}\ (\Rightarrow \neg\vee) \qquad \frac{\Gamma \Rightarrow \neg\varphi\{y/x\}, \Delta}{\Gamma \Rightarrow \neg\exists x\varphi, \Delta}\ (\Rightarrow \neg\exists)$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge \Rightarrow) \qquad \frac{\Gamma, \varphi\{t/x\} \Rightarrow \Delta}{\Gamma, \forall x\varphi \Rightarrow \Delta}\ (\forall \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \neg\varphi, \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \wedge \psi), \Delta}\ (\Rightarrow \neg\wedge) \qquad \frac{\Gamma \Rightarrow \neg\varphi\{t/x\}, \Delta}{\Gamma \Rightarrow \neg\forall x\varphi, \Delta}\ (\Rightarrow \neg\forall)$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}\ (\Rightarrow \wedge) \qquad \frac{\Gamma \Rightarrow \varphi\{y/x\}, \Delta}{\Gamma \Rightarrow \forall x\varphi, \Delta}\ (\Rightarrow \forall)$$

Fig. 1. The system $\mathbf{QG^4_{EIP}}$ in standard form

Theorem 2. $\mathbf{QG^4_{EIP}}$ *admits strong cut-elimination, and* $\mathcal{QM}_E^4$ *is strongly characteristic for it.*

Proof. One can mechanically check that $\mathbf{QG^4_{EIP}}$ is coherent (e.g. see Example 3). It follows from Theorem 1 that $\mathbf{QG^4_{EIP}}$ admits strong cut-elimination and that $\mathcal{M}_{\mathbf{QG^4_{EIP}}}$ is characteristic for it. It remains to show that $\mathcal{QM}_E^4 = \mathcal{M}_{\mathbf{QG^4_{EIP}}}$.

As an example, consider a non-empty prefix X of $\mathbb{N}$ and a function $h : X \to \mathcal{V}$ with image $\{\perp, \mathrm{t}\}$. In $\mathcal{QM}_E^4$ one has $\tilde{\forall}_X [h] = \{\perp, \mathrm{f}\}$. For $\mathcal{M}_{\mathbf{QG}^4_{\mathbf{EIP}}}$ one must find which $\forall$-rules of $\mathbf{QG}^4_{\mathbf{EIP}}$ are members of $R_{\mathbf{QG}^4_{\mathbf{EIP}}} [\forall, X, h]$. Let $\mathcal{N}$ be a L_1^1-structure for $\mathcal{M}_4$ such that $\mathsf{Dom}\, \mathcal{N} = X$ and $p_1{}^{\mathcal{N}} = h$. Pick $\xi_\perp \in h^{-1} [\perp]$ and $\xi_\mathrm{t} \in h^{-1} [\mathrm{t}]$. Consider each $\forall$-rule of $\mathbf{QG}^4_{\mathbf{EIP}}$:

- If $c_1{}^{\mathcal{N}} = \xi_\mathrm{t}$, then $p_1{}^{\mathcal{N}} \left[c_1{}^{\mathcal{N}}\right] = \mathrm{t}$, and so $\mathcal{N} \models \{p_1 (c_1) \Rightarrow\}$.
  Thus $\{p_1 (c_1) \Rightarrow\} / (\forall \Rightarrow) \in R_{\mathbf{QG}^4_{\mathbf{EIP}}} [\forall, X, h]$.
- There exists an $\mathcal{N}$-substitution τ such that $(\tau [v_1])^{\mathcal{N}} = \xi_\perp$, so $\mathcal{N} \not\models \{\Rightarrow p_1 (v_1)\}$.
  Thus $\{\Rightarrow p_1 (v_1)\} / (\Rightarrow \forall) \notin R_{\mathbf{QG}^4_{\mathbf{EIP}}} [\forall, X, h]$.
- Note that $p_1{}^{\mathcal{N}} \left[c_1{}^{\mathcal{N}}\right] \in \{\mathrm{t}, \perp\}$, so $\tilde{\neg}^4 p_1{}^{\mathcal{N}} \left[c_1{}^{\mathcal{N}}\right] \in \{\mathrm{f}, \perp\}$, and so $\mathcal{N} \not\models \{\Rightarrow \neg p_1 (c_1)\}$.
  Thus $\{\Rightarrow \neg p_1 (c_1)\} / (\Rightarrow \neg\forall) \notin R_{\mathbf{QG}^4_{\mathbf{EIP}}} [\forall, X, h]$.

Therefore, in $\mathcal{M}_{\mathbf{QG}^4_{\mathbf{EIP}}}$, $\tilde{\forall}_X [h] = \bigcap \{F [\{p_1 (c_1) \Rightarrow\} / (\forall \Rightarrow)]\} = \{\mathrm{f}, \perp\}$.

The other cases are similar.

Conclusion & Future Research

We have shown that for a very wide class of quasi-canonical Gentzen-type proof systems, our syntactic criterion of coherence is equivalent to both strong cut-elimination and to strong soundness and completeness. Hence the task of proving cut-elimination (which is often rather difficult) now becomes very easy for systems in this class, since it involves only the trivial matter of verifying the coherence criterion. Using this result we extended the framework of Existential Information Processing to predicate logic with dual-arity quantifiers. Parallelizing the propositional case, non-deterministic semantics and a strongly sound and complete proof system were given for this extension, and the admissibility of strong cut-elimination for that system was shown.

There are several directions of further research following this paper.

- Including function symbols in the schematic representation language(s) from Definition 15 (not just constants) to express explicit dependencies between variables and terms in the application forms of canonical rules.
- Definition 20 only addresses systems in which there are no rules of type $(\neg \Rightarrow)$ or $(\Rightarrow \neg)$, however in [1] systems with one such rule (of a specific shape) are also considered, yielding systems for three-valued logics.[6] These systems require a bit more care in their analysis (c.f. [1, Definition 5.5] of $\bar{x}$-inconsistency where $x \in \{\mathrm{f}, \mathrm{t}, \top, \perp\}$). Still, we expect such 3-quasi-canonical systems could similarly be extended to first-order logic.
- Theorem 1 may be seen as evidence that canonicity is a flexible concept, and so similar theorems may be provable for other kinds of Gentzen-type proof systems. The systems dealt with in [9] and [10] are natural candidates.

[6] The addition of more than one such rule is uninteresting as it result in a system that is either trivial or equivalent to a (non-quasi) canonical one.

- The existential strategy is just one possible information gathering strategy. A more interesting one involves a reservoir equipped with an order indicating authority. This enables the *authoritative* strategy, in which information is gathered only from sources that have not been overruled by a superior one.
- Sources in a reservoir share the same algebra. This means they are all aware of the same individuals, and agree about the meaning of all function symbols. A generalization which captures situations where this is not the case would be interesting, and increase the usefulness of this framework.
- Formulas that are classically equivalent are not equivalent in this framework. For example, a source may assign 1 to $\varphi \vee (\psi \wedge \theta)$ yet assign 0 to $(\varphi \vee \psi) \wedge (\varphi \vee \theta)$. The issue is in mitigating this with minimal complications.

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